

Harmonic oscillator on noncommutative spaces

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A generalized harmonic oscillator on noncommutative spaces is considered. Dynamical symmetries and physical equivalence of noncommutative systems with the same energy spectrum are investigated and discussed. General solutions of three-dimensional noncommutative harmonic oscillator are found and classified according to dynamical symmetries. We have found conditions under which three-dimensional noncommutative harmonic oscillator can be represented by ordinary, isotropic harmonic oscillator in effective magnetic field.

I. INTRODUCTION

Recently, it has been realized that noncommutative geometry plays a distinguished role in the formulation of string theory [1, 2] and M-theory [3]. It has been shown that, in a certain limit, the entire string dynamics can be described by minimally coupled gauge theory on noncommutative space [2]. Noncommutative field theory [4] has been constructed by introducing the Moyal product in the space of ordinary functions, and by defining field theory in quantum phase space. Equivalence between these two approaches has been clarified in Ref.[5].

Presumably, noncommutative effects are important at very high energies. Nevertheless, we could observe high-energy effects in the low-energy effective action, or we could use noncommutative geometry in constructing the low-energy effective action. For example, the noncommutative Chern-Simons gauge theory, represented as a matrix theory of elementary charges, has been used to describe quantum Hall physics [6]. Phenomenological aspects of spacetime noncommutativity have been analyzed using a noncommutative extension of the standard model [7]. Recently, it was proposed to use synchrotron radiation to search for experimental evidence of the effects of noncommutativity [8]. It has turned out that in many proposals to test the hypothetical spacetime noncommutativity, it is sufficient to consider only quantum mechanical approximation [9, 10].

Subsequently, quantum mechanics on noncommutative spaces has been extensively studied [11–27]. In Ref.[28] two of us have presented a unified approach to representations of noncommutative quantum mechanics in arbitrary dimensions and have given conditions for physical equivalence of different representations. Also, we have shown that there exist two physically distinct phases in arbitrary dimensions that are connected by a discrete duality transformation. Furthermore, we have discussed symmetries in phase space and the dynamical symmetry of a physical system, and shown how these symmetries are affected by change in commutation relations.

Throughout the paper we assume that the time coordinate is commutative, since otherwise we would be forced to modify the usual scheme of quantum mechanics [29]. We analyze a noncommutative harmonic oscillator (ho) in detail, especially three-dimensional case. Various deformations of the harmonic oscillator have been discussed in the literature, including the ho in the quantum group framework [30], the ho with minimal length uncertainty relations [31], the ho in noncommutative spaces [12, 16–18], the (super)ho on CP^N [32], just to mention the latest. The overcomplete symmetry of the ho enables one to exactly solve the oscillator problem even in some deformed cases. Especially, the ho is the only exactly solvable model in noncommutative quantum mechanics. As for applications, the ho represents a prototype of a physical system in every branch of physics, but the most exciting applications are probably Bose-Einstein condensation and quantum Hall effect in various dimensions.

Surprisingly, the simplest physically relevant system, the three-dimensional ho, have not yet been analyzed in detail. There exist some results concerning very special choice of noncommutative parameters [16, 18, 20]. The most general parametrization of the noncommutative three-dimensional ho is nontrivial extension of the two-dimensional case, what can be best seen from our analysis of the dynamical symmetry of this noncommutative system.

The plan of the paper is the following. In Section II we will introduce the general formalism used in the paper. Then, in Section III we will present some results concerning harmonic oscillator on noncommutative space in arbitrary dimensions. The main results of the paper are related to the case of three-dimensional harmonic oscillator and are

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presented in Sect. IV. There, we have presented conditions under which three-dimensional noncommutative harmonic oscillator can be represented by ordinary, isotropic harmonic oscillator in the effective magnetic field, see Refs.[12, 17] for two-dimensional case. And finally, in the last section we summarize our results.

The conventions we use in this paper are the following: summation over repeated indices is assumed, small latin indices i, j, \dots go over configuration space dimensions, capital latin indices I, J, \dots go over phase space dimensions, bold face fonts denote vectors, prime denotes physical quantity in phase II of parameter space.

II. NONCOMMUTATIVE SPACES IN ARBITRARY DIMENSIONS

Let us define D -dimensional noncommutative coordinate operators X_1, \dots, X_D and the corresponding noncommutative momentum operators P_1, \dots, P_D with commutation relations

$$\begin{aligned} [X_i, X_j] &= i\theta_{ij}, \quad [P_i, P_j] = iB_{ij}, \quad i, j = 1, \dots, D \\ [X_i, P_i] &= ih_i, \quad [X_i, P_j] = i \begin{cases} \phi_{ij} & i < j \\ -\psi_{ij} & i > j \end{cases} \end{aligned} \quad (1)$$

and X_i, P_i are hermitean operators, $X_i^\dagger = X_i, P_i^\dagger = P_i$. The $2D$ - dimensional noncommutative phase space is described by variables $(U_1, U_2, \dots, U_{2D}) = (X_1, P_1, \dots, X_D, P_D)$ satisfying

$$\begin{aligned} [U_I, U_J] &= iM_{IJ}, \quad U_I^\dagger = U_I, \quad I, J = 1, \dots, 2D, \\ M_{JI} &= -M_{IJ}, \quad (M_{IJ})^\dagger = M_{IJ}. \end{aligned} \quad (2)$$

Generally, M_{IJ} is an operator depending on U_I . However the Jacobi identities $[U_I, [U_J, U_K]] + \text{cycl.} = 0$ restrict the choice of the operators M_{IJ} . An important physical requirement is that in the limit $\theta_{ij}, B_{ij}, \phi_{ij}, \psi_{ij} \rightarrow 0$ and $h_i \rightarrow \hbar$, we should obtain canonical variables u_I of ordinary quantum mechanics in a smooth manner.

If we start with an arbitrary real, nonlinear, and regular (invertible) mapping $U_I = U_I(u_J)$ and $u_J = u_J(U_I)$ connecting noncommuting and commuting phase-space variables, we obtain $[U_I, U_J] = if_{IJ}(u_K) = iM_{IJ}(U_L)$ satisfying all the above restrictions including the Jacobi identities. The reverse is not true globally. Namely, by postulating the matrix M satisfying the Jacobi identities it is not clear whether the above mentioned mapping exists. However, one can start from Eq.(2) and find a *local* Darboux transformation $U_I = U_I(u_J)$, and $u_J = u_J(U_I)$.

The simplest examples of noncommutative spaces are i) $[X_i, X_j] = i\theta_{ij}$, with θ_{ij} c-numbers, ii) $[X_i, X_j] = ic_{ijk}X_k$, Lie-algebra type, iii) $[X_i, P_j] = i[\delta_{ij}(1 + \beta\mathbf{P}^2) + \beta'P_iP_j]$, introducing minimal length uncertainty relations [31], iv) $X_iX_j = R_{ij,kl}X_kX_l$, for example, the Manin plane, etc. Interesting and important physical questions are: i) What are the symmetries of such spaces and the corresponding conservation laws?, ii) How can one define classical and quantum physics on such spaces in a consistent way?, iii) What are physical consequences of noncommutativity, i.e., what are deviations from physics on ordinary spaces? iv) What are physical applications and the role of singular spaces (example - noncommutative Landau problem)?

In the rest of the paper we concentrate on a simple noncommutative space with c-number commutators M_{IJ} , Eq.(2). Note that $\det M \geq 0$. Singular spaces (spaces at the critical point in parameter space) are defined by $\det M = 0$, and regular (nondegenerate) spaces have $\det M$ strictly positive. The antisymmetric, real matrix M is defined by $D(2D - 1)$ parameters. It can be brought to a universal, block-diagonal form

$$R^T M R = \begin{pmatrix} 0 & \omega_1 & & & \\ -\omega_1 & 0 & & & \\ & & 0 & \omega_2 & \\ & & -\omega_2 & 0 & \\ & & & & \ddots \end{pmatrix}, \quad (3)$$

where R is an orthogonal matrix with $\det R = 1$, and $\pm\omega$'s are real numbers, eigenvalues of the matrix (iM) . The characteristic equation is of order $2D$ and contains only even powers in ω :

$$\prod_{i=1}^D (\omega^2 - \omega_i^2) = 0.$$

The product of eigenvalues $\kappa = \prod \omega_i > 0$ defines phase I, smoothly connected with ordinary space, and $\kappa < 0$ defines phase II in parameter space. In phase I we choose all eigenvalues positive, and in phase II we choose $\omega_D < 0$

and eigenvalues $\omega_1, \omega_2, \dots, \omega_{D-1}$ positive. The connection between the two phases is established by flip F in the phase-space variables $X_D \leftrightarrow P_D$, i.e., $\omega_D \leftrightarrow -\omega_D$:

$$F = \begin{pmatrix} 1 & 0 & & & \\ 0 & 1 & & & \\ & & \ddots & & \\ & & & 1 & 0 \\ & & & 0 & 1 \\ & & & & 0 & 1 \\ & & & & 1 & 0 \end{pmatrix}. \quad (4)$$

We define the duality relations between the two phases, $|\omega_i| = |\omega'_i|$, $\kappa = -\kappa'$ [28]. These relations connect the physical systems with $M' = R'FR^TMRFR'^T$, with $R, R' \in SO(2D)$. A singular space is characterized by the degree of the eigenvalue $\omega = 0$.

In the rest of the paper we assume $\omega_i \neq 0$. The transformation R , Eq.(3), defines new variables $U_I^0 = R_{IJ}^T U_J$, i.e., (X_i^0, P_i^0) with the commutator

$$[X_i^0, P_j^0] = i\omega_i \delta_{ij},$$

and all other commutators being zero. We can further transform the variables to ordinary canonical ones $u_I = (D^{-1})_{IJ} U_J^0$, or $u_I = (FD^{-1})_{IJ} U_J^0$ in phase II, where the matrix D is $D = \text{diag}(\sqrt{\omega_1}, \sqrt{\omega_1}, \dots, \sqrt{\omega_{D-1}}, \sqrt{\omega_{D-1}}, \sqrt{|\omega_D|}, \sqrt{|\omega_D|})$:

$$\left. \begin{aligned} x_i &= X_i^0 / \sqrt{\omega_i}, \quad p_i = P_i^0 / \sqrt{\omega_i}, \quad \forall i, \quad \text{in phase I} \\ x_i &= X_i^0 / \sqrt{\omega_i}, \quad p_i = P_i^0 / \sqrt{\omega_i}, \quad i = 1, \dots, D-1, \\ x_D &= P_D^0 / \sqrt{|\omega_D|}, \quad p_D = X_D^0 / \sqrt{|\omega_D|} \end{aligned} \right\} \text{in phase II.} \quad (5)$$

The transformation RD (RDF) connecting the initial noncommuting coordinates U_I with the canonical u_I is invertible but not unitary.

Note that the matrix M , Eq.(2), is invariant under a group of transformations isomorphic to $Sp(2D)$. Furthermore, the orthogonal matrix R is unique up to the orthogonal transformations preserving M ($S^T M S = M$).

In D -dimensions, angular momentum operators are generators of coordinate space rotations, preserving $\mathbf{X}^2 = \sum_i X_i^2$, $\mathbf{P}^2 = \sum_i P_i^2$, $\mathbf{X}\mathbf{P} = X_i P_i$:

$$\begin{aligned} [J_{ij}, X_k] &= i(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})X_l, \\ [J_{ij}, P_k] &= i(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk})P_l, \quad i, j, k, l = 1, \dots, D, \end{aligned}$$

and generally,

$$[J_{ij}, U_K] = i(E_{ij})_{KL} U_L, \quad K, L = 1, \dots, 2D.$$

For a regular matrix M , we can construct the angular momentum generators $J_{ij} = -\frac{1}{2}(E_{ij}M^{-1})_{KL}U_KU_L$ only if $[E_{ij}, M] = 0$, for all $i, j = 1, \dots, D$. In $D = 2$ case the angular momentum J can be constructed only if $h_1 = h_2 = h$ and $\psi = \phi$

$$J = \frac{1}{h^2 - \theta B + \phi^2} \left\{ h(X_1 P_2 - X_2 P_1) + \frac{B}{2}(X_1^2 + X_2^2) + \frac{\theta}{2}(P_1^2 + P_2^2) - \phi(X_1 P_1 + X_2 P_2) \right\}. \quad (6)$$

For $D \geq 3$ there is no $SO(D)$ symmetry and we cannot construct all $D(D-1)/2$ angular momentum generators J_{ij} . There is at most $\left[\frac{D}{2}\right]$ generators of rotations in mutually commuting noncommutative planes.

III. ISOTROPIC NONCOMMUTATIVE OSCILLATOR IN ARBITRARY DIMENSIONS

The system is defined by the Hamiltonian

$$H = \frac{1}{2} \sum_{i=1}^D (P_i^2 + X_i^2) = \frac{1}{2} \sum_{I=1}^{2D} U_I^2, \quad (7)$$

where the constants \hbar , m , and ω are absorbed in phase-space variables. The Hamiltonian (7) is invariant under $O(2D)$ transformations. The commutation relations are described by the c-numbers M_{IJ} . We represent this system in terms of the canonical variables u_I , Eq.(5),

$$H = \frac{1}{2} \sum_{I=1}^{2D} U_I^{0\ 2} = \left\{ \begin{array}{l} \frac{1}{2} \sum_{I=1}^{2D} (Du)_I^2 \quad \text{phase I} \\ \frac{1}{2} \sum_{I=1}^{2D} (DFu)_I^2 \quad \text{phase II} \end{array} \right\} = \frac{1}{2} \sum_{i=1}^D |\omega_i| (p_i^2 + x_i^2). \quad (8)$$

The matrix F , Eq.(4), represents a discrete transformation connecting the two phases. The energy spectrum is

$$E_{n_1 \dots n_D} = \sum_{i=1}^D |\omega_i| \left(n_i + \frac{1}{2} \right), \quad n_i \in N_0. \quad (9)$$

The Hamiltonian (8) and the energy spectrum (9) are equal in both phases, for all M with the same eigenvalues $|\omega_i|$. They correspond to the anisotropic oscillator in canonical variables. The initial isotropic noncommutative oscillator is not unitarily equivalent to the anisotropic oscillator in canonical variables, but all corresponding physical quantities of noncommutative harmonic oscillator can be uniquely determined. Only representations connected by transformations preserving the commutation relations are physically equivalent.

The degenerate energy levels for the Hamiltonian (7) are described by a set of orthogonal eigenstates that transform according to an irreducible representation of the dynamical symmetry group. The dynamical symmetry group $G(H, M)$ is a group of all transformations preserving both, commutation relations M and the Hamiltonian H . For the fixed Hamiltonian, the dynamical symmetry depends on M ; so, by changing the parameters of the matrix M we can change $G(H, M)$ from $G_{min}(H, M)$ to $G_{max}(H, M)$. For the noncommutative harmonic oscillator, the minimal dynamical symmetry group is $[U(1)]^D$ and the maximal symmetry group is $U(D)$. Hence, different choices of M correspond to different dynamical symmetry. This can be viewed as a new mechanism of symmetry breaking with the origin in (phase)space structure. The underlying theory that would determine M does not exist yet.

The generators of dynamical symmetry are quadratic in phase space variables, i.e., of the type $\mathcal{G} = C_{IJ} U_I U_J$, where the real coefficients C_{IJ} can be chosen to be symmetric, $C_{IJ} = C_{JI}$. The generators can be determined as null-eigenvectors of the matrix $A_{IJ, KL}$:

$$[H, U_I U_J] = \sum_{K, L} A_{IJ, KL} U_K U_L = \frac{1}{2} \sum_K [M_{KI} (U_J U_K + U_K U_J) + M_{KJ} (U_I U_K + U_K U_I)].$$

For the two-dimensional harmonic oscillator, we have calculated [28] the generators of $U(2)$ and generic $U(1) \times U(1)$ symmetry. There are additional non-symplectic symmetries if ω_i/ω_j are rational number [33].

A. $U(D)$ dynamical symmetry

As we have already mention, the maximal dynamical symmetry group for the noncommutative harmonic oscillator is $U(D)$, and in that case the matrix M has all eigenvalues identical, $\omega_i = \omega$. The general conditions are:

$$M^T M = \omega^2 \cdot \mathbb{I}_{2D \times 2D}, \quad M^T = -M. \quad (10)$$

In two dimensions we have solved Eqs.(10), and obtained the following conditions on noncommutativity parameters resulting in $U(2)$ symmetry of noncommutative harmonic oscillator in two dimensions [28]:

$$\begin{aligned} h_1 &= h_2 = h, \quad \theta = -B, \quad \phi = \psi, \quad \text{in phase I,} \\ h_1 &= -h_2, \quad \theta = B, \quad \phi = -\psi, \quad \text{in phase II.} \end{aligned} \quad (11)$$

The angular momentum J exists in phase I

$$J = \frac{1}{h^2 + \theta^2 + \phi^2} \left[h(X_1 P_2 - X_2 P_1) - \phi(X_1 P_1 + X_2 P_2) - \frac{\theta}{2}(X_1^2 + X_2^2 - P_1^2 - P_2^2) \right],$$

but does not exist in phase II.

For $D = 4, 5, 6$, we can write a simple family of antisymmetric matrices M that provide maximal $U(D)$ dynamical symmetry, with fixed eigenvalue ω . If $h_i = h$ for $i = 1, \dots, D$, then

$$M_4 = \begin{pmatrix} J_0 & A & B & A^T \\ -A^T & J_0 & A & B \\ -B & -A^T & J_0 & A \\ -A & -B & -A^T & J_0 \end{pmatrix}, \quad M_5 = \begin{pmatrix} J_0 & A & B & B & A^T \\ -A^T & J_0 & A & B & B \\ -B & -A^T & J_0 & A & B \\ -B & -B & -A^T & J_0 & A \\ -A & -B & -B & -A^T & J_0 \end{pmatrix}, \quad (12)$$

and

$$M_6 = \begin{pmatrix} J_0 & A & B & B & B & A^T \\ -A^T & J_0 & A & B & B & B \\ -B & -A^T & J_0 & A & B & B \\ -B & -B & -A^T & J_0 & A & B \\ -B & -B & -B & -A^T & J_0 & A \\ -A & -B & -B & -B & -A^T & J_0 \end{pmatrix}, \quad (13)$$

where

$$J_0 = \begin{pmatrix} 0 & h \\ -h & 0 \end{pmatrix}, \quad A = \begin{pmatrix} a & b \\ c & -a \end{pmatrix}, \quad B = \begin{pmatrix} d & e \\ e & -d \end{pmatrix}. \quad (14)$$

The fixed eigenvalue is

$$\omega^2 = h^2 + 2a^2 + b^2 + c^2 + (D-3)(d^2 + e^2). \quad (15)$$

The real parameters a, b, c, d, e, h satisfy the following relations:

$$2ad + h(b-c) + e(b+c) + (D-4)(d^2 + e^2) = 0, \text{ for } D = 4, 5, 6, \quad (16)$$

and in addition

$$a^2 + bc + h(b-c) + d^2 + e^2 = 0, \text{ for } D = 5, 6. \quad (17)$$

The most general case for $D = 3$ is discussed in the next section. Eqs.(12) and (13) represent the parametrization in phase I, since there exists a smooth limit to ordinary quantum mechanics. The dual solution in phase II is FMF , and is obtained using the flip transformation (4). More generally, there are solutions of Eqs.(10) with h_i mutually different.

The noncommutative ho with $U(D)$ dynamical symmetry can be represented by ordinary isotropic ho. These systems are not physically equivalent although they have the same energy spectrum. We can calculate physical quantities of interest for ordinary ho, and, using the mapping RD (RDF), determine (uniquely) corresponding physical quantities for noncommutative ho. For harmonic oscillator defined on noncommutative space with $D \geq 3$, there is no angular momentum generators.

IV. HARMONIC OSCILLATOR IN THREE DIMENSIONS

The three-dimensional harmonic oscillator is the simplest, physically relevant system. The most general matrix M is

$$M = \begin{pmatrix} 0 & h_1 & \theta_3 & \phi_3 & -\theta_2 & -\phi_2 \\ -h_1 & 0 & \psi_3 & B_3 & -\psi_2 & -B_2 \\ -\theta_3 & -\psi_3 & 0 & h_2 & \theta_1 & \phi_1 \\ -\phi_3 & -B_3 & -h_2 & 0 & \psi_1 & B_1 \\ \theta_2 & \psi_2 & -\theta_1 & -\psi_1 & 0 & h_3 \\ \phi_2 & B_2 & -\phi_1 & -B_1 & -h_3 & 0 \end{pmatrix}, \quad (18)$$

with $\theta_{ij} = \varepsilon_{ijk}\theta_k$, and similarly for other parameters, $B_{ij}, \phi_{ij}, \psi_{ij}$, defined in Eq.(1). The critical points are determined by

$$\det M = \left[h_1 h_2 h_3 + \sum_i h_i (\phi_i \psi_i - \theta_i B_i) + \phi_1 \phi_3 \psi_2 - \phi_2 \psi_1 \psi_3 \right. \quad (19)$$

$$\left. + B_1 (\psi_3 \theta_2 - \psi_2 \theta_3) + B_2 (\psi_1 \theta_3 - \phi_3 \theta_1) + B_3 (\phi_2 \theta_1 - \phi_1 \theta_2) \right]^2 = 0. \quad (20)$$

The eigenvalues of the matrix iM are obtained from

$$\omega^6 - \alpha\omega^4 + \beta\omega^2 - \gamma = 0, \quad (21)$$

where

$$\begin{aligned} \alpha &= \omega_1^2 + \omega_2^2 + \omega_3^2 = -\frac{1}{2}\text{Tr}M^2, \\ \beta &= \omega_1^2\omega_2^2 + \omega_2^2\omega_3^2 + \omega_1^2\omega_3^2 = \frac{1}{8}(\text{Tr}M^2)^2 - \frac{1}{4}\text{Tr}M^4, \\ \gamma &= \omega_1^2\omega_2^2\omega_3^2 = \det M. \end{aligned} \quad (22)$$

Phase I is defined by $\kappa = \omega_1\omega_2\omega_3 > 0$ and all ω 's positive, and phase II is defined by $\kappa < 0$ and $\omega_{1,2} > 0, \omega_3 < 0$. The duality relations connecting the two phases with the same energy spectrum are $\omega_1 = \omega'_1, \omega_2 = \omega'_2, \omega_3 = -\omega'_3$. Singular spaces are characterized by i) $\omega_1 > 0$ and $\omega_2 = \omega_3 = 0$ and ii) $\omega_1 \geq \omega_2$ and $\omega_3 = 0$.

The spectrum of the noncommutative oscillator is the same as for the anisotropic harmonic oscillator, see Eqs.(7) and (8):

$$E_{n_1 n_2 n_3} = \omega_1 \left(n_1 + \frac{1}{2} \right) + \omega_2 \left(n_2 + \frac{1}{2} \right) + |\omega_3| \left(n_3 + \frac{1}{2} \right), \quad n_1, n_2, n_3 \in N_0. \quad (23)$$

All physical quantities for the noncommutative oscillator can be uniquely calculated knowing the transformation RD , i.e., RDF .

The generic dynamical symmetry is $[U(1)]^3$ when all eigenvalues $|\omega_i|$ are mutually different. When two eigenvalues are the same, up to a sign, $\omega_1 = \omega_2 \neq \omega_3$ or $\omega_1 \neq \omega_2 = \pm\omega_3$, the dynamical symmetry is $U(2) \times U(1)$. The sign of ω_3 determines the phase. In the special case $\omega_1 = \omega_2 = \pm\omega_3$, we have the $U(3)$ symmetry group.

A. $U(3)$ dynamical symmetry

In the case with $U(3)$ dynamical symmetry, $M^T M = \omega^2 \cdot \mathbb{I}_{6 \times 6}$ and the most general parametrization of commutation relations with $h_i = h$, leading to $U(3)$ symmetry is

$$M = \begin{pmatrix} J_0 & A & A^T R \\ -A^T & J_0 & \varepsilon AR \\ -R^T A & -\varepsilon R^T A^T & J_0 \end{pmatrix}, \quad (24)$$

where the matrices J_0 and A are given by Eq.(14), $\varepsilon^2 = 1$, and

$$R = \begin{pmatrix} \cos \alpha & -\sin \alpha \\ \sin \alpha & \cos \alpha \end{pmatrix}. \quad (25)$$

The real parameters a, b, c, h satisfy the following condition:

$$h(b - c) + \varepsilon(a^2 + bc) = 0,$$

and the common eigenvalue is $\omega^2 = h^2 + 2a^2 + b^2 + c^2$. The matrix M , Eq.(24), belongs to phase I. In phase II, $U(3)$ dynamical symmetry can be parametrized by the matrix FMF , where F is the flip matrix, Eq.(4). More generally, there are solutions of Eqs.(10) with h_i mutually different. For example, we can have

$$M = \begin{pmatrix} 0 & h & \theta & \phi & 0 & 0 \\ -h & 0 & \phi & -\theta & 0 & 0 \\ -\theta & -\phi & 0 & h & 0 & 0 \\ -\phi & \theta & -h & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & h_3 \\ 0 & 0 & 0 & 0 & -h_3 & 0 \end{pmatrix}, \quad (26)$$

with $\omega^2 = h_3^2 = h^2 + \theta^2 + \phi^2$, also leading to the $U(3)$ dynamical symmetry.

We have seen, Eqs.(24) and (26), that there is a class of noncommutative isotropic oscillators with $U(3)$ dynamical symmetry that are physically different. All of them are described by the same Hamiltonian $H = \sum U_I^2/2$ and possess

the same energy spectrum. They are connected by orthogonal transformations R (RF). However, they are not unitarily equivalent, unless $[M, R] = 0$. The easiest way to see the difference is to consider the possibility of saturation of uncertainty relations [23, 28], since $\Delta U_I \Delta U_J \geq |M_{IJ}|/2$. Especially interesting is the case $M^T M = \mathbb{I}_{6 \times 6}$, where noncommutative ho $H = \sum U_I^2/2$ and ordinary isotropic ho $H = \sum u_I^2/2$ have the same form, the identical spectrum, but have different matrix elements of observables. In the latter case we can construct angular momentum operators, whereas in the former (noncommutative) case this is not possible.

A special class of $U(3)$ invariant systems has been proposed in Ref.[20], in order to retain $U(3)$ symmetry of ordinary isotropic ho. They have started with the harmonic oscillator in terms of canonical variables $H = \sum u_I^2/2$ and then transformed the system by the nonunitary transformation $U = R D u$, thus obtaining $H = \sum (D^{-1} R^{-1} U)_I^2/2$. This system is quadratic in U_J , but is not diagonal, and possesses $U(3)$ symmetry by construction. But this is just one special case in the large class of $U(3)$ -symmetric noncommutative harmonic oscillators that we have described in detail.

B. Simple extension of two-dimensional ho

There is a simple parametrization of the matrix (18), i.e., an extension of two-dimensional ho. Imagine we have noncommutative plane commuting with the rest of the space. In that case the matrix M is block-diagonal. There is one 4×4 block representing four-dimensional phase space (X_1, P_1, X_2, P_2) of noncommutative plane, and one 2×2 block of the remaining coordinate (X_3, P_3) . We choose $B_i, \theta_i, \phi_i, \psi_i = 0$, for $i = 1, 2$. This reduces to the most general two-dimensional harmonic oscillator [28]. Specially, eigenvalues of the matrix iM are:

$$\begin{aligned} \omega_{1,2} &= \frac{1}{2} \sqrt{(\theta - B)^2 + (\phi + \psi)^2 + (h_1 + h_2)^2} \pm \frac{1}{2} \sqrt{(\theta + B)^2 + (\phi - \psi)^2 + (h_1 - h_2)^2}, \\ \omega_3 &= h_3, \quad \kappa = (h_1 h_2 - \theta B + \phi \psi) h_3. \end{aligned} \quad (27)$$

The phases are determined by the sign of $\kappa = \omega_1 \omega_2 \omega_3$. The duality relations between system with the same energy spectrum in the two phases were constructed in Ref.[28].

We can always represent noncommutative ho in terms of anisotropic ho in commuting coordinates. But, in two dimensions we can also represent the two-dimensional anisotropic oscillator as the two-dimensional isotropic oscillator in the effective magnetic field, using a symplectic transformation between canonical variables

$$H = \frac{1}{2} \sum_{i=1,2} [(p_i - A_i)^2 + \omega_{\text{eff}}^2 x_i^2] = \frac{1}{2} \sum_{i=1,2} [p_i^2 + (\omega_{\text{eff}}^2 + B_{\text{eff}}^2) x_i^2] - \frac{1}{2} B_{\text{eff}} (x_1 p_2 - x_2 p_1), \quad (28)$$

where $\mathbf{A} = (-x_2, x_1, 0) B_{\text{eff}}/2$. In phase I, effective frequency and effective magnetic field are, for general two-dimensional case:

$$\omega_{\text{eff}}^2 = \omega_1 \omega_2, \quad B_{\text{eff}} = \omega_1 - \omega_2 = \sqrt{(\theta + B)^2 + (\phi - \psi)^2 + (h_1 - h_2)^2}. \quad (29)$$

In phase II the corresponding physical quantities are

$$\omega_{\text{eff}}^2 = \omega_1 |\omega_2|, \quad B_{\text{eff}} = \omega_1 - |\omega_2| = \sqrt{(\theta' - B')^2 + (\phi' + \psi')^2 + (h'_1 + h'_2)^2}. \quad (30)$$

This physical interpretation is possible only in two dimensions, or if noncommutative plane decouples from the rest of the higher-dimensional space. The condition for the three-dimensional isotropic oscillator in the effective magnetic field along the third axis, $B_{\text{eff}} = \omega_1 - |\omega_2|$, is $\omega_1 |\omega_2| = \omega_3^2 = \omega_{\text{eff}}^2$.

Note that all three types of dynamical symmetry, i.e., $U(3)$, $U(2) \times U(1)$, and $[U(1)]^3$ are possible.

C. Special parametrization in terms of Θ and \mathbf{B}

For the rest of this section we choose $h_i = 1$ and $\phi_{ij} = \psi_{ij} = 0$, i.e., we impose $[X_i, P_j] = i\delta_{ij}$. We organize remaining parameters in the general matrix (18) in two vectors, $\Theta = (\theta_1, \theta_2, \theta_3)$ and $\mathbf{B} = (B_1, B_2, B_3)$. The condition for a critical point is $\kappa = 1 - \Theta \mathbf{B} = 0$, and the sign of κ determines the phase. The coefficients in the characteristic equation (21) are

$$\begin{aligned} \alpha &= 3 + \Theta^2 + \mathbf{B}^2 \geq 3, \\ \beta &= 3 + \Theta^2 \mathbf{B}^2 + (\Theta - \mathbf{B})^2 \geq 3, \\ \gamma &= (1 - \Theta \mathbf{B})^2 \geq 0. \end{aligned} \quad (31)$$

The coefficients α, β, γ , and therefore the eigenvalues ω_i , are invariant under rotation in three-dimensional space, $\theta'_i = R_{ij}\theta_j$, $B'_i = R_{ij}B_j$, $R \in O(3)$.

At the critical point, $\gamma = 0$, i.e., $\Theta\mathbf{B} = 1$. The equations (31) imply $\alpha^2 - 4\beta > 5$. There is only one zero, $\omega_3 = 0$. The two remaining eigenvalues are

$$\omega_{1,2}^2 = \frac{\alpha \pm \sqrt{\alpha^2 - 4\beta}}{2}.$$

At the critical point, the dynamical symmetry is $[U(1)]^3$.

D. Θ and \mathbf{B} collinear

Especially interesting is the case when Θ and \mathbf{B} are collinear and $\phi_{ij} = \psi_{ij} = 0$. Then $\kappa = \omega_1\omega_2\omega_3 = 1 - \Theta\mathbf{B}$, and the eigenvalues ω_i can be reduced to the two-dimensional problem in the plane orthogonal to Θ . Namely, if we choose $\Theta = (0, 0, \theta)$ and $\mathbf{B} = (0, 0, B)$, we find a solution of characteristic equation (21):

$$\omega_{1,2} = \sqrt{\left(\frac{\theta - B}{2}\right)^2 + 1} \pm \left|\frac{\theta + B}{2}\right|, \quad \omega_3 = 1. \quad (32)$$

The solution can belong to phases I, II or to the singular case, depending on $\kappa = \omega_1\omega_2\omega_3$.

The matrix R , defined in Eq.(3), can be written in this parametrization in the following form:

$$R = \begin{pmatrix} \cos \varphi & 0 & \sin \varphi & 0 & 0 & 0 \\ 0 & \sin \varphi & 0 & \cos \varphi & 0 & 0 \\ 0 & \cos \varphi & 0 & -\sin \varphi & 0 & 0 \\ -\sin \varphi & 0 & \cos \varphi & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \quad (33)$$

where we choose $\varphi \in (0, \pi/2)$, $\theta \geq 0$, $\theta + B \geq 0$ and

$$\cos \varphi = \frac{1}{\sqrt{1 + (B + \omega_2)^2}} = \frac{\omega_2 + \theta}{\sqrt{1 + (\omega_2 + \theta)^2}} = \sqrt{\frac{\omega_1 - B}{\omega_1 + \omega_2}}. \quad (34)$$

The anisotropic oscillator with energy spectrum Eq.(32) can be represented as isotropic oscillator in (x_1, x_2) plane in magnetic field orthogonal to that plane, see Eqs.(29) and (30). In given parametrization, $h_1 = h_2 = 1$, Eq.(29) reduces to [18, 25]

$$\begin{aligned} \omega_{\text{eff}}^2 &= 1 - \theta B, \quad B_{\text{eff}} = \theta + B, \quad \text{in phase I,} \\ \omega_{\text{eff}}^2 &= \theta' B' - 1, \quad B_{\text{eff}} = \sqrt{(\theta' - B')^2 + 4}, \quad \text{in phase II.} \end{aligned} \quad (35)$$

Note that in the case $B = 0$ (or $\theta = 0$) we have three-dimensional isotropic ho in effective magnetic field $B_{\text{eff}} = \theta$ (or $B_{\text{eff}} = B$), since $\omega_{\text{eff}}^2 = 1 = \omega_3^2$.

Some simple examples are the following:

a) if $\mathbf{B} = 0$, the result belongs to phase I,

$$\omega_{1,2} = \sqrt{1 + \frac{\theta^2}{4}} \pm \frac{\theta}{2}, \quad \omega_3 = 1.$$

We can construct symplectic transformation connecting anisotropic oscillator with 3D-isotropic oscillator in magnetic field with effective frequency and effective magnetic field

$$\omega_{\text{eff}}^2 = 1, \quad B_{\text{eff}} = \theta.$$

b) the choice $\theta = B$ leads to

$$\omega_{1,2} = 1 \pm \theta, \quad \omega_3 = 1,$$

and it is singular for $\theta = 1$, it belongs to phase I for $\theta < 1$, and to phase II for $\theta > 1$. The effective frequency and effective magnetic field of isotropic oscillator in magnetic field are

$$\omega_{\text{eff}}^2 = 1 - \theta^2, \quad B_{\text{eff}} = 2\theta < 2 \text{ in phase I,}$$

and

$$\omega_{\text{eff}}^2 = \theta^2 - 1, \quad B_{\text{eff}} = 2 \text{ in phase II,}$$

Also, for $\theta = B = 2$, the eigenvalues are $\omega_1 = 3, \omega_2 = -1, \omega_3 = 1$ and we have $U(2) \times U(1)$ dynamical symmetry in phase II.

c) choosing $\theta = -B$ give us $U(2) \times U(1)$ dynamical symmetry in phase I, since

$$\omega_1 = \omega_2 = \sqrt{1 + \theta^2}, \quad \omega_3 = 1.$$

In this case, effective magnetic field cancels, $B_{\text{eff}} = 0$, and we have isotropic ho in two dimensions with effective frequency $\omega_{\text{eff}}^2 = 1 + \theta^2$. The generators of the dynamical symmetry group were found in Ref.[28].

An interesting physical example is the noncommutative Landau problem [12, 24]. In two dimensions it can be represented as a noncommutative harmonic oscillator with $\omega \rightarrow 0$ and also as a noncommutative harmonic oscillator with $\tilde{\omega} \neq 0$, at the critical point $\tilde{\theta}\tilde{B} = 1$. The connection between parameters is $\tilde{\omega}^2\tilde{\theta} + 1/\tilde{\theta} = B$.

With parametrization chosen in this subsection, the angular momentum in the conventional sense (see the definition (6)) can be defined only in the plane orthogonal to Θ :

$$J_{12} = J_z = \frac{1}{1 - \theta B} \left[X_1 P_2 - X_2 P_1 + \frac{B}{2}(X_1^2 + X_2^2) + \frac{\theta}{2}(P_1^2 + P_2^2) \right]. \quad (36)$$

Analyzing the characteristic equation we find the following statements concerning dynamical symmetries with nontrivial noncommutative parameters ($\theta^2 + B^2 > 0$).

i) There is no $U(3)$ dynamical symmetry in this parametrization. Namely, from $\omega_1 = 1$ it follows that $\theta = B$ and $\theta = -B$, and this is possible only for $\theta = B = 0$.

ii) $U(2) \times U(1)$ dynamical symmetry can be realized in three ways.

First, there is a case described under c) above. There we have $\omega_1 = \omega_2 > 1$, $\omega_3 = 1$ and Θ and \mathbf{B} are antiparallel; Secondly, we can have the following case in phase I:

$$\omega_1 = 1 + \frac{\theta^2}{\theta + 1} > 1, \quad \omega_2 = \omega_3 = 1.$$

For $\theta > 0$ and $\theta + B > 0$,

$$B = -\frac{\theta}{\theta + 1},$$

i.e., \mathbf{B} is antiparallel to Θ . The effective frequency of two-dimensional isotropic ho is $\omega_{\text{eff}} = \omega_1$ and effective magnetic field is $B_{\text{eff}} = \theta^2/(\theta + 1)$;

Finally, in phase II, for $\theta > 1$ and $\theta + B > 0$, we have

$$\omega_1 = \frac{\theta^3 + 1}{\theta^2 - 1} > 1, \quad \omega_2 = -1, \quad \omega_3 = 1.$$

\mathbf{B} and Θ are parallel and

$$B = \frac{\theta}{\theta - 1}.$$

The effective frequency of two-dimensional isotropic ho is $\omega_{\text{eff}} = \omega_1$ and effective magnetic field is $B_{\text{eff}} = \theta^2/(\theta - 1) - 2$. Note that effective magnetic field B_{eff} need not be zero in order to have $U(2) \times U(1)$ dynamical symmetry. Also note that in above examples $\omega_{\text{eff}} > 1$, hence there is no representation in terms of three-dimensional isotropic ho in effective magnetic field.

iii) in all other cases the generic dynamical symmetry is $[U(1)]^3$.

E. Arbitrary position of Θ and \mathbf{B}

In the more general case, when Θ and \mathbf{B} are not collinear, the characteristic equation (21) leads to solutions that are not easy to analyze. However, it is remarkable that the above statements for the dynamical symmetry group hold even when we extend the analysis to the case when Θ and \mathbf{B} are not collinear.

Proposition: Let us assume that the noncommuting coordinates and momenta satisfy

$$[X_i, X_j] = i\varepsilon_{ijk}\theta_k, \quad [P_i, P_j] = i\varepsilon_{ijk}B_k, \quad [X_i, P_j] = i\delta_{ij}, \quad (37)$$

where θ_i and B_i are real c-numbers, $\theta^2 + B^2 > 0$, and consider the three-dimensional noncommutative isotropic oscillator (7). Then, $U(2) \times U(1)$ dynamical symmetry is possible if and only if \mathbf{B} and Θ are collinear, in three special cases. In all other cases, the generic dynamical symmetry is $[U(1)]^3$.

Proof: Let us express the invariants $\theta^2 + B^2$, $\theta^2 B^2$, and $\Theta\mathbf{B}$ using α, β , Eq.(22), and κ :

$$\begin{aligned} \theta^2 + B^2 &= \alpha - 3, \\ \theta^2 B^2 &= \beta - \alpha + 2 - 2\kappa, \\ \Theta\mathbf{B} &= 1 - \kappa. \end{aligned} \quad (38)$$

$U(2) \times U(1)$ dynamical symmetry implies that two out of three eigenvalues are equal, up to a sign, say $\omega_1^2 = \omega_2^2 = \omega^2 \neq \omega_3^2$. Inserting $\omega_1^2 = \omega_2^2 = \omega^2$ into (22), and using the inequalities

$$\theta^2 + B^2 > 0, \quad (\Theta\mathbf{B})^2 \leq \theta^2 B^2 \leq \left(\frac{\theta^2 + B^2}{2}\right)^2, \quad (39)$$

we find the following inequalities:

From $\theta^2 + B^2 > 0$ it follows

$$2\omega^2 + \omega_3^2 > 3.$$

From $(\Theta\mathbf{B})^2 \leq \theta^2 B^2$ it follows

$$(\omega_3^2 - 1)(\omega^2 - 1)^2 \leq 0.$$

From $4\theta^2 B^2 \leq (\theta^2 + B^2)^2$ it follows

$$(\omega_3 - 1)^2[(\omega_3 + 1)^2 - 4\omega^2] \geq 0.$$

If $\omega_3 = 1$, then $\omega^2 > 1$, and if $\omega^2 = 1$, then $\omega_3^2 > 1$. If $\omega_3 \neq 1$ and $\omega^2 \neq 1$, the above inequalities lead to a contradiction. From the above analysis it follows that all $\omega_1^2 = \omega_2^2$ solutions are possible if and only if $(\Theta\mathbf{B})^2 = \theta^2 B^2$, i. e., when \mathbf{B} and Θ are collinear. When \mathbf{B} and Θ are collinear, there are three possible realizations of $U(2) \times U(1)$ symmetry as has already been shown in Subsection C. $U(3)$ symmetry is not possible for this parametrization with $\theta^2 + B^2 > 0$.

In conclusion, with parametrization $[X_i, P_j] = i\delta_{ij}$, only if \mathbf{B} and Θ are collinear, $\omega_3 = 1$ and it is possible to represent noncommutative ho as ordinary $2D$ isotropic oscillator in effective magnetic field. But, noncommutative ho can be represented as $3D$ ordinary isotropic oscillator in effective magnetic field $B_{\text{eff}} = \omega_1 - |\omega_2|$ even when \mathbf{B} and Θ are not collinear, provided $\omega_1|\omega_2| = \omega_3^2$. This condition is sufficient even for the most general parametrization of commutation relations M , Eq.(18).

F. Application - Nilsson model

As we have already noticed, different choices of M correspond to different dynamical symmetries and this can be interpreted as a new mechanism of symmetry breaking with the origin in phase space structure. There are possible applications to bound states in atomic, nuclear, and particle physics. Let us briefly discuss the nuclear shell model, more specifically, the Nilsson model. The basis of the shell model in a finite nucleus is the assumption of an independent particle motion within a mean field, and therefore the nuclear Hamiltonian can be written as a sum of single-particle Hamiltonians over all active nucleons [34]. The central binding potential for a nucleon may be approximated by the harmonic oscillator well and in this case, the single particle Hamiltonian is described by isotropic three-dimensional ho. Phenomenological improvement of such a simple model is done by breaking the rotational symmetry of harmonic well, leading to anisotropic ho with frequencies fitted from experiment.

We suggest that starting from the simple Hamiltonian (7) defined on noncommutative space with $[X_i, P_j] = i\delta_{ij}$, we can formulate an effective theory that would account for the spectrum of low-lying excitations of a nucleus. The parameters of the matrix M , Eq.(18), can be interpreted as parameters describing the background, namely, the mean field acting on a single nucleon. For example, we have shown in subsection D that the spectrum $e = \hbar\omega_1(n_1 + 1/2) + \hbar\omega_2(n_2 + 1)$ can be obtained from (7) when Θ and \mathbf{B} are collinear. In this case, noncommutativity parameters have a clear physical interpretation, representing the preferred direction, thus breaking rotational invariance down to axial invariance. We have seen that we can define angular momentum in the plane orthogonal to Θ , Eq.(36), so for particles with spin there is an additional term in Hamiltonian (7), $H_{\text{int}} \sim \mathbf{J} \cdot \mathbf{s}$. There is a correction of order θ in comparison to commutative case. For large deformations we use the general parametrization of the matrix M , Eq.(18). The limit $\Theta, \mathbf{B} \rightarrow 0$ reproduces rotationally symmetric, isotropic oscillator. This being the effective theory, the parameters Θ, \mathbf{B} should be also determined from experimental results. The difference between our proposal and standard Nilsson model should be in the transition amplitudes. Namely, we can fit the noncommutative parameters to have the desired energy spectrum (same as in commutative effective theory), but then the matrix elements of observables would be different.

V. DISCUSSION AND OUTLOOK

We have considered a noncommutative, $O(2D)$ symmetric oscillator with constant (c-number) commutation relations M_{ij} in $2D$ -dimensional nonsingular ($\det M \neq 0$) phase-space. There exist two, physically distinct, phases, defined by $\kappa = \prod \omega_i \gtrless 0$. If the matrix M is block-diagonal ($\det M = \prod \det M_i$), then it is characterized by the set of κ_i , corresponding to the subspaces. A discrete duality transformation connects two systems with the same energy spectra in two different phases. We have presented a unified approach for analyzing a noncommutative ho in arbitrary dimensions and both phases simultaneously. General construction of transformations from noncommutative variables to canonical Darboux variables is presented. Starting from Hamiltonian $H(U)$ and commutation relations M and applying linear transformations, we obtain different physical systems with the same energy spectrum and the same dynamical symmetry (up to isomorphism). Namely, the noncommutative ho is mapped to an ordinary, commutative anisotropic ho with the same spectrum as a noncommutative ho. Since the transformation RD (RDF) is not unitary, these two systems are not physically equivalent, although all physical quantities can be uniquely determined. These systems differ in matrix elements of observables, uncertainty relations and other physical properties. Only systems with the same energy spectra and the same commutation relations are physically (unitarily) equivalent.

The dynamical symmetry of the noncommutative isotropic oscillator is $\prod U(D_i)$, $\sum D_i = D$. We have presented a family of matrices M leading to the maximal $U(D)$ symmetry of the ho in D -dimensional noncommutative space, for $D < 7$. The noncommutative ho with $U(D)$ dynamical symmetry can be represented by ordinary isotropic ho.

We have presented a detailed analysis of the three-dimensional noncommutative ho. Our main result is the parametrization of the matrix M for different dynamical symmetry groups, $U(3)$, $U(2) \times U(1)$, $[U(1)]^3$. Especially, the most general conditions for maximal symmetry $U(3)$ are presented. We have shown that for a special parametrization of commutation relations, $h_i = 1$, $\phi_{ij} = \psi_{ij} = 0$, there is no $U(3)$ symmetry. Furthermore, the $U(2) \times U(1)$ dynamical symmetry is possible if and only if Θ and \mathbf{B} are collinear (only in three particular cases). For an arbitrary angle, different from zero and π , between the vectors Θ and \mathbf{B} , the dynamical symmetry is $[U(1)]^3$.

We have found generally that three-dimensional noncommutative harmonic oscillator can be represented by ordinary, $2D$ isotropic harmonic oscillator in effective magnetic field only in noncommutative plane that commutes with the third dimension, or by $3D$ isotropic ho in the effective magnetic field provided $\omega_1|\omega_2| = \omega_3^2$. Angular momentum operators in noncommutative spaces can be defined only as generators of rotations in noncommutative planes which mutually commute. The physical interpretation of noncommutative effects in quantum mechanics in higher dimensions is not yet clear and requires further investigations.

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